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A geometrical approach to $N = 2$ super Yang-Mills theory on the two dimensional lattice

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Abstract: We propose a discretization of two dimensional euclidean Yang-Mills theories with $N = 2$ supersymmetry which preserves exactly both gauge invariance and an element of supersymmetry. The approach starts from the twisted form of the continuum super Yang Mills action which we show may be written in terms of two real Kähler-Dirac fields whose components transform into each other under the twisted supersymmetry. Once the theory is written in this geometrical language it is straightforward to discretize by mapping the component tensor fields to appropriate geometrical structures in the lattice and by replacing the continuum exterior derivative and its adjoint by appropriate lattice covariant difference operators. The lattice action is local and possesses a unique vacuum state while the use of Kähler-Dirac fermions ensures the model does not exhibit spectrum doubling.

Keywords: Topological Field Theories, Lattice Gauge Field Theories, Extended Supersymmetry.

Contents

1. Introduction

Supersymmetric field theories play a central role in modern theories of particle physics. From a phenomenological viewpoint they are attractive as providing a solution to the gauge hierarchy problem [1]. From a theoretical perspective they are more tractable analytically than their non-supersymmetric counterparts while still exhibiting features like confinement and chiral symmetry breaking [2]. Super Yang-Mills theories are especially interesting because of their possible connection to string and M-theory [3].

For these reasons a good deal of effort has gone into attempts to formulate such theories on spacetime lattices see, for example, [4, 5] and the recent reviews by Feo and Kaplan [6, 7]. However, until recently these efforts mostly met with only limited success. The reasons for this are well known — generic discretizations of supersymmetric field theories break supersymmetry at the classical level leading to the appearance of a plethora of *relevant* SUSY breaking counterterms in the effective action. The couplings to all these terms must then be fine tuned as the lattice spacing is reduced in order that the theory approach a supersymmetric continuum limit. This problem is particularly acute in theories with extended supersymmetry which contain scalar fields.

One might hope that this fine tuning problem might be reduced or perhaps even eliminated by formulating the lattice models in such a way as to preserve some element of SUSY on the lattice. An approach following this philosophy has been described in papers by Kaplan et al. [8, 9, 10].

In [11] we proposed a different scheme, useful for theories with extended supersymmetry, based on a reformulation of the theories using ideas drawn from topological quantum field theory. The key to this approach is to construct a new rotation group from a combination of the original rotation group and part of the R-symmetry associated with the extended SUSY. The supersymmetric field theory is then reformulated in terms of fields which transform as integer spin representations of this new rotation group [12]. This process is given the name *twisting* and in flat space one can think of it as merely an exotic change of variables in the theory. In this process a scalar anticommuting field is always produced associated with a nilpotent supercharge Q. Furthermore, as argued in [13] the twisted superalgebra implies that the action rewritten in terms of these twisted fields is generically Q-exact. In this case it is straightforward to construct a lattice action which is Q-invariant provided *only* that we preserve the nilpotency of Q under discretization. Concrete examples of this construction for theories without gauge symmetries were given in [14, 15, 16] corresponding to supersymmetric quantum mechanics, the 2D complex Wess-Zumino model and supersymmetric sigma models.

In [17] the conditions allowing for a nilpotent supercharge were analyzed in some detail within a conventional superspace approach. In [18] a twisted superspace formalism was developed and used to construct models which preserved all the twisted supercharges at the expense of introducing some non-commutativity at the scale of lattice spacing. In [19] Sugino managed to extend the technique of latticization via twisting to the case of models with gauge symmetry by a non-trivial modification of the twisted supersymmetry transformations. However, the lattice models constructed this way have some difficulties — they generically suffer from a vacuum degeneracy problem[1] and the lattice actions are not rotationally invariant. Both of these problems may be traced to the requirement that all fields *except* for the gauge links transform identically under the gauge group despite their differing spins and hence geometrical characters. We are thus motivated to seek a more geometrical approach to discretization of the continuum twisted theory.

In this paper we propose an alternative lattice regularization scheme for two-dimensional $N = 2$ (euclidean) super Yang-Mills theory. Our jumping off point is again the continuum twisted theory. First, to show that these twisted models are completely equivalent in flat space to the conventional formulations, we show how to reconstruct the usual super Yang-Mills theory written in terms of spinor fields from the twisted model. Next we introduce the notion of a Kähler-Dirac field and recall the relationship between the Kähler-Dirac equation and the usual Dirac equation. We show that the anticommuting twisted fermion fields arising in the super Yang-Mills model are nothing more than components of a single real Kähler-Dirac field. The usual flavor index of the Kähler-Dirac field is now naturally associated with an index describing the behavior of the field under additional R-symmetries associated with the extended supersymmetry. This construction yields an explicit example of the connection between twisting and the Kähler-Dirac fermion mechanism emphasized in recent papers [21, 22, 18]. The connection to Kähler-Dirac fermions is important as it has been known for some time how to discretize the latter equation without encountering spectrum doubling [23]–[26]. Indeed, we show that the twisted lattice fermion action we propose is nothing more than a latticized, gauged Kähler-Dirac action for fields in the adjoint representation of the gauge group.

[1] This problem was circumvented in the case of $N = 2$ SYM in two dimensions [20].

Furthermore, we can show that the entire theory can be recast as one involving a single Kähler-Dirac field with grassmann components representing the twisted fermions, together with another Kähler-Dirac field with commuting component fields representing the scalars, auxiliary field and gauge field. The twisted supersymmetry operator then induces transformations between corresponding components of these Kähler-Dirac fields. This fully geometric representation of the continuum theory can then be naturally discretized while preserving gauge invariance, supersymmetry and without inducing fermion doubles. The discretization prescription we use was first proposed in [27] and maps continuum fields which transform differently under the (twisted) rotation group to different geometrical features in the hypercubic lattice. Specifically we assign scalar fields to sites, vector fields to links, rank 2 antisymmetric tensor fields to plaquettes etc. These fields will then be taken to transform *differently* at finite lattice spacing under gauge transformations. In addition we will introduce two covariant finite difference operators which are compatible with these differing gauge transformation properties of the fields. They will represent the lattice analogs of the exterior derivative and its adjoint. Using these ingredients we will show that it is rather straightforward to latticize the continuum twisted theory while maintaining both invariance under lattice gauge transformations and a single twisted supersymmetry. The resultant action is moreover local, has a unique vacuum state and is free of doubler modes.

2. Two dimensional continuum $N = 2$ SYM

Our starting point will be the continuum twisted form of the two dimensional $N = 2$ SYM model which possesses two scalar fields $\phi, \overline{\phi}$, a vector A_μ and another commuting field B_{12} corresponding to the single independent component of a rank 2 antisymmetric tensor field in two dimensions. The fermions of the theory appear as an anticommuting scalar field η, a vector ψ_μ and a field χ_{12} conjugate to B_{12}. All these fields are taken in the adjoint representation of some gauge group $C = \sum_a T^a C^a$ where the T^a's will be taken to be *anti-hermitian* generators of the group and the component fields C^a are real. The twisted action takes the form

$$S = \beta Q \,\mathrm{Tr} \int d^2x \left(\frac{1}{4}\eta[\phi, \overline{\phi}] + 2\chi_{12}F_{12} + \chi_{12}B_{12} + \psi_\mu D_\mu \overline{\phi} \right) \tag{2.1}$$

where the object inside the Q-variation we shall refer to as the twisted gauge fermion in analogy with usual BRST terminology. The twisted supersymmetry acts on the fields as

$$\begin{aligned} QA_\mu &= \psi_\mu \\ Q\psi_\mu &= D_\mu \phi \\ Q\phi &= 0 \\ Q\chi_{12} &= B_{12} \\ QB_{12} &= [\phi, \chi_{12}] \\ Q\overline{\phi} &= \eta \\ Q\eta &= [\phi, \overline{\phi}]\,. \end{aligned} \tag{2.2}$$

Notice that the square of twisted supersymmetry operator yields an infinitesimal gauge transformation $Q^2 = \delta_G^\phi$ with parameter ϕ. Carrying out the Q-variation and integrating out the multiplier field B_{12} leads to the form

$$S = \beta \operatorname{Tr} \int d^2x \left(\frac{1}{4}[\phi, \overline{\phi}]^2 - \frac{1}{4}\eta[\phi, \eta] - F_{12}^2 + D_\mu \phi D_\mu \overline{\phi} \right.$$
$$\left. - \chi_{12}[\phi, \chi_{12}] - 2\chi_{12}(D_1\psi_2 - D_2\psi_1) - \psi_\mu D_\mu \eta + \psi_\mu[\overline{\phi}, \psi_\mu] \right). \quad (2.3)$$

The coefficient of F_{12}^2 appears negative but this is an illusion. With our representation of the generators $\operatorname{Tr}\{T_a, T_b\} = -\delta_{ab}$ and the gauge action written in terms of component fields is positive semidefinite. To show that this twisted model is nothing more than the usual SYM theory, in which the fermions are represented by spinor fields, we construct a Dirac spinor out the four (real) anticommuting twisted fields

$$\Psi = \begin{pmatrix} \frac{1}{2}\eta - i\chi_{12} \\ \psi_1 - i\psi_2 \end{pmatrix}. \quad (2.4)$$

It is straightforward to see that the kinetic terms in 2.3 can be rewritten in the Dirac form

$$\Psi^\dagger \gamma \cdot D \Psi \quad (2.5)$$

where the gamma matrices are taken in the euclidean chiral representation

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (2.6)$$

In the same way the Yukawa interactions with the scalar fields can be written

$$\Psi^\dagger \frac{(1+\gamma_5)}{2}[\overline{\phi}, \Psi] - \Psi^\dagger \frac{(1-\gamma_5)}{2}[\phi, \Psi] \quad (2.7)$$

where γ_5 in this representation is

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.8)$$

Thus the on-shell twisted action is nothing more than the usual $N = 2$ SYM action in two dimensions. Notice that to make this correspondence and obtain a bounded euclidean action it is necessary to think of ϕ and $\overline{\phi}$ as complex conjugates rather than real independent fields. In the continuum theory this complexification is a little mysterious but we will see that it is natural within a lattice framework. Finally with ϕ and $\overline{\phi}$ complex conjugates it is easy to show that $\gamma_1 M^* \gamma_1 = M$ where M is the fermion operator which implies that the fermion determinant is (generically) positive definite.

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3. Interpretation in terms Kähler-Dirac fields

The fact that the fermions of the twisted model are represented by (antisymmetric) tensor fields is reminiscent of the component fields entering into the Kähler-Dirac equation. In this section we verify this connection by showing how to write the original twisted SYM theory entirely in terms of such Kähler-Dirac fields. We start by recalling the properties of the Kähler-Dirac equation and its connection to spinor fields and the usual Dirac equation [23, 25, 26]

In D dimensions we introduce a Kähler-Dirac field ω whose components are antisymmetric tensor fields or p-forms where $p = 0 \ldots D$. Thus

$$\omega = (f, f_\mu, f_{\mu\nu}, \ldots)\,. \tag{3.1}$$

We can define the action of the exterior derivative d on such a field by the action on its components

$$d\omega = (0, \partial_\mu f, \partial_\mu f_\nu - \partial_\nu f_\mu, \ldots) \tag{3.2}$$

A natural dot product between two such Kähler-Dirac fields A and B is given by

$$\langle A|B\rangle = \int d^D x \sqrt{g} \sum_p A^{\mu_1 \ldots \mu_p} B_{\mu_1 \ldots \mu_p}\,. \tag{3.3}$$

The adjoint of the exterior derivative $d^\dagger$ can then be defined in terms of the component fields as

$$-d^\dagger \omega = \left(f^\nu, f^\nu_\mu, \ldots, 0\right)_{;\nu}\,. \tag{3.4}$$

If we form the matrix (from now on we will assume flat euclidean space)

$$\Psi^{(\omega)}_{\alpha\beta}(x) = \sum_{p=0}^{D} (\gamma^{\mu_1} \cdots \gamma^{\mu_p})_{\alpha\beta}\, f_{\mu_1 \ldots \mu_p} \tag{3.5}$$

it is straightforward to show that that the following equation

$$\gamma^\mu_{\alpha\alpha'} \partial_\mu \Psi^{(\omega)}_{\alpha'\beta} = 0 \tag{3.6}$$

is equivalent to the Kähler-Dirac equation

$$(d - d^\dagger)\omega = 0\,. \tag{3.7}$$

Furthermore we can interpret eq. 3.6 as the usual Dirac operator acting on a multiplet of $2^{\frac{D}{2}}$ identical flavors of Dirac fermions labeled by the index β. This statement is unaffected if gauge interactions are introduced and the derivative operator d replaced by an appropriate gauge covariant exterior derivative D. Furthermore, the Kähler-Dirac equation can be derived from an action which can be written in two equivalent ways

$$\begin{aligned} S &= \frac{1}{2} \mathrm{Tr}\, \Psi^\dagger \gamma_\mu \partial_\mu \Psi \\ &= \langle \omega^\dagger | (d - d^\dagger)\omega \rangle \end{aligned}$$

where the (anticommuting) matrix valued fields Ψ and $\Psi^\dagger$ (and correspondingly the Kähler-Dirac fields ω, $\omega^\dagger$) are to be treated as independent fields. To make contact with the twisted SYM model discussed earlier let us examine in detail the case of two dimensions. Choosing

$$\omega = \left(\frac{\eta}{2}, \psi_\mu, \chi_{12}\right) \tag{3.8}$$

we find the the continuum Kähler-Dirac action when expanded on the component fields yields

$$S = \int d^2x \left(\frac{1}{2}\eta^\dagger D_\mu \psi_\mu + \frac{1}{2}\psi_\mu^\dagger D_\mu \eta + \psi_2^\dagger D_1 \chi_{12} - \psi_1^\dagger D_2 \chi_{12} + \chi_{12}^\dagger \left(D_1 \psi_2 - D_2 \psi_1\right)\right). \tag{3.9}$$

If we now impose the condition that the component fields are purely anti-hermitian (or hermitian) we obtain the continuum twisted $N = 2$ action examined earlier (up to an unimportant factor of minus two). Notice that in this case the total number of real fields needed to write down the Kähler-Dirac model (four) exactly matches the number of real supercharges of the $N = 2$ theory in two dimensions. We thus see that the fermionic sector of the $N = 2$ model in two dimensions is naturally given in terms of a single real Kähler-Dirac fermion. It is clear that the bosonic sector of the model contains a similar set of four real fields $\overline{\phi}$, A_μ and B_{12} together with the gauge degree of freedom ϕ. Therefore let us introduce another Kähler-Dirac field $\Phi = (\overline{\phi} - \phi, A_\mu, B_{12})$ with commuting components. Consider the following expression

$$\langle \Psi | \left(d_A \Phi + Q\Psi\right)\rangle \tag{3.10}$$

where d_A denotes the covariant form of d. Writing out components yields

$$\int d^2x \left(\psi_\mu D_\mu \overline{\phi} + \chi_{12} 2F_{12} + \frac{1}{4}\eta[\phi, \overline{\phi}] + \chi_{12} B_{12}\right). \tag{3.11}$$

This is nothing more than the gauge fermion used in the continuum twisted SYM model. Actually we should be a little careful here — to derive the correct fermionic Kähler-Dirac action we should really introduce two independent Kähler-Dirac fields Ψ and $\Psi^\dagger$. This necessitates using a gauge fermion of the form

$$\frac{1}{2}\langle \Psi^\dagger | (d_A \Phi + Q\Psi)\rangle + \frac{1}{2}\langle (d_A \Phi + Q\Psi)^\dagger | \Psi\rangle\,. \tag{3.12}$$

There are two ways to reduce this theory to the usual Yang-Mills model. We have already seen one simple method — assume we can impose a reality condition on the fields after Q-variation. This is what we shall do later in the lattice theory. However, in the continuum there is another way to proceed by requiring that the fields Ψ and $\Psi^\dagger$ appearing in the gauge fermion eq. 3.12 be replaced by *self-dual* fields Ψ_+ and $\Psi_+^\dagger$ where

$$\Psi_+ = P_+ \Psi \tag{3.13}$$

and the projection operator is given by $P_+ = \frac{I+*}{2}$. The $*$ symbol denotes a duality operation (related to the Hodge dual) taking p forms into $(D-p)$ forms. For a p-form A the associated dual $(D-p)$-form has components

$$A_{\nu_1 \ldots \nu_{D-p}} = i(-1)^{\frac{1}{2}(D-p)(D-p+1)} \epsilon_{\mu_1 \ldots \mu_p | \nu_1 \ldots \nu_{D-p}|} A_{\mu_1 \ldots \mu_p} \tag{3.14}$$

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where the notation $|\mu_1 \ldots \mu_p|$ means only terms with $\mu_1 < \mu_2 < \cdots \mu_p$ are included in the sum. The tensor ϵ is the completely antisymmetric symbol in D dimensions. In the matrix language the projection corresponds to right multiplication by the matrix $\frac{(1+\gamma_5)}{2}$. The resulting Kähler-Dirac matrices contain a single non-zero column corresponding to a single Dirac spinor and the reduction is complete.

The above analysis shows that the usual twisted SYM theory can be elegantly rewritten in the language of Kähler-Dirac fields. This rewriting of the theory in terms of differential forms has two primary advantages — it allows us to formulate the theory on a curved space and also gives a natural starting point for discretization. Indeed it has been shown [23, 25, 24] that any non-gauge theory formulated in such geometrical terms may be discretized on a hypercubic lattice *without inducing fermion doubling* by replacing the exterior derivative d by a forward difference operator D^+_μ and its adjoint $d^\dagger$ by a backward difference $-D^-_\mu$. Furthermore, in [27], it was shown how to construct covariant versions of these difference operators for fields taking their values in the adjoint representation of a gauge group. It is hence natural to try to use Kähler-Dirac fields to formulate lattice supersymmetric actions. Attempts were made to construct such theories in a hamiltonian formalism in [28, 29]. A similar approach was used in [30] to construct a SYM model in euclidean space using only some of the component Kähler-Dirac fields. However it is only in the context of twisted supersymmetry that the full power of the Kähler-Dirac approach can be realized.

4. General prescription for discretization

We list here for reference the essential ingredients in our discretization prescription. Notice they do not depend on dimension.

- A continuum p-form field $f_{\mu_1\ldots\mu_p}(x)$ will be mapped to a corresponding lattice p-form field associated with the p-dimensional hypercube at lattice site x spanned by the (positively directed) unit vectors $\{\mu_1 \ldots \mu_p\}$.
- Such a lattice field will transform under gauge transformations[2] in the following way
$$f_{\mu_1\ldots\mu_p}(x) \rightarrow G(x) f_{\mu_1\ldots\mu_p}(x) G^{-1}(x + e_{\mu_1\ldots\mu_p}) \qquad (4.1)$$
where the vector $e_{\mu_1\ldots\mu_p} = \sum_{j=1}^{p} \mu_j$.
- To construct gauge invariant quantities we will need to introduce both $f_{\mu_1\ldots\mu_p}$ and its hermitian conjugate $f^\dagger_{\mu_1\ldots\mu_p}(x)$. The latter transforms as
$$f^\dagger_{\mu_1\ldots\mu_p}(x) \rightarrow G(x + e_{\mu_1\ldots\mu_p}) f_{\mu_1\ldots\mu_p}(x) G^{-1}(x) \qquad (4.2)$$
This differing transformation law for the field and its adjoint requires that the component fields $f^a_{\mu_1\ldots\mu_p}$ be treated as complex. This complexification of the degrees of freedom can be extended to scalar fields *provided* they are required to be (anti)self-conjugate $f^\dagger = -f$. Notice that such a definition departs from the usual notion of

[2] We use G^{-1} rather than $G^\dagger$ to allow us to consider complexified gauge transformations later — we thank Joel Giedt for this suggestion.

hermitian conjugation but is natural if we want to consider a theory with a complexified gauge invariance.

- For a continuum gauge field we introduce lattice link fields $U_\mu(x) = e^{A_\mu(x)}$ and its conjugate $U_\mu^\dagger = e^{A_\mu^\dagger(x)}$.

- A covariant forward difference operator can be defined which acts on a field $f_{\mu_1\ldots\mu_p}(x)$ as follows

$$D_\mu^+ f_{\mu_1\ldots\mu_p}(x) = U_\mu(x) f_{\mu_1\ldots\mu_p}(x+\mu) - f_{\mu_1\ldots\mu_p}(x) U_\mu(x + e_{\mu_1\ldots\mu_p})\,. \tag{4.3}$$

  This operator acts like a lattice exterior derivative with respect to gauge transformations in mapping a p-form lattice field to a $(p+1)$-form lattice field.

- Similarly we can define an adjoint operator D_μ^- whose action on some field $f_{\mu_1\ldots\mu_p}$ is given by

$$D_\mu^- f_{\mu_1\ldots\mu_p}(x) = f_{\mu_1\ldots\mu_p}(x) U_\mu^\dagger(x + e_{\mu_1\ldots\mu_p} - \mu) - U_\mu^\dagger(x-\mu) f_{\mu_1\ldots\mu_p}(x-\mu) \tag{4.4}$$

- To discretize the continuum theory formulated in geometrical language simply map all p-form fields to lattice fields as described above and replace all instances of d by D^+ and $d^\dagger$ by D^-.

- In the final path integral we choose a contour along on which the imaginary part of the gauge field is zero and such that the action is real, positive definite.

5. Two dimensional lattice $N = 2$ SYM

We start from an expression for the lattice gauge fermion which is identical to the continuum one eq. 3.12

$$\frac{1}{2}\langle \Psi^\dagger | (D\Phi + Q\Psi)\rangle + \frac{1}{2}\langle (D\Phi + Q\Psi)^\dagger | \Psi \rangle \tag{5.1}$$

where, following our discretization prescription, a lattice Kähler-Dirac field is composed of (complex-valued) p-form fields defined on p-dimensional hypercubes in the lattice

$$\begin{aligned}\Phi &= \left(\overline{\phi} - \phi, U_\mu, B_{12}\right)\\ \Psi &= \left(\frac{1}{2}\eta, \psi_\mu, \chi_{12}\right)\end{aligned}$$

together with the conjugate fields $\Phi^\dagger$ and $\Psi^\dagger$. The fields possess the gauge transformation properties listed in the previous section. Thus a site, link and plaquette field transform under a gauge transformation $G(x) = e^{\phi(x)}$ as

$$\begin{aligned}f(x) &\to G(x) f(x) G^{-1}(x)\\ f_\mu(x) &\to G(x) f_\mu(x) G^{-1}(x+\mu)\\ f_{\mu\nu}(x) &\to G(x) f_{\mu\nu}(x) G^{-1}(x+\mu+\nu)\end{aligned}$$

while their conjugates transform in the complementary way

$$\begin{aligned} f^\dagger(x) &\to G(x)f(x)G^{-1}(x) \\ f^\dagger_\mu(x) &\to G(x+\mu)f_\mu(x)G^{-1}(x) \\ f^\dagger_{\mu\nu}(x) &\to G(x+\mu+\nu)f_{\mu\nu}(x)G^{-1}(x)\,. \end{aligned}$$

While we are free to regard the site fields ϕ, $\overline{\phi}$ and η as complex-valued the above transformations require them to satisfy an (anti)self-conjugacy condition eg. $\overline{\phi}^\dagger = -\overline{\phi}$. Notice that this requirement is consistent with the promotion of the original gauge invariance to invariance under the complexified group — the lattice gauge transformation $G^\dagger = G^{-1}$ if $\phi^\dagger = -\phi$. It is clear that these transformations reduce to the usual ones for connections and fields in the adjoint of the gauge group in the naive continuum limit. As usual we regard the gauge link as the exponential of some matrix $U_\mu(x) = e^{A_\mu(x)}$ where $A_\mu(x)$ and indeed all other lattice fields may be expanded on a basis of (anti-hermitian) traceless generators of the gauge group $f(x) = \sum_a f^a(x)T^a$. The doubling of degrees of freedom in the lattice theory is, at first sight, a little puzzling — clearly in the fermionic sector it is nothing more than the usual statement that the spinors ψ and $\overline{\psi}$ are to considered as independent in euclidean space. However, in a model with twisted supersymmetry this necessarily seems to imply a corresponding doubling of bosonic states. Another way to understand this doubling of p-form fields is to recognize that it can be taken to represent the two possible orientations of the underlying p-dimensional hypercube. The complexification of the vector potential $A^a_\mu(x)$ has the benefit of allowing the fields $U(x)$ and $U^\dagger(x)$ to vary independently under the twisted supersymmetry. In the end we will require the final path integral be taken along a contour where $U^\dagger U = I$ and the imaginary parts of the gauge field and the fermion fields vanish. This reality condition will allow contact to be made with the usual twisted continuum theory.

Returning now to the expression for the lattice gauge fermion in Kähler-Diraclanguage we define the lattice covariant exterior derivative D acting on Kähler-Dirac fields in terms of the action of D^+_μ on the component fields

$$D\Phi = \left(0, D^+_\mu\left(\overline{\phi}-\phi\right), 2\mathcal{F}_{12}\right) \tag{5.2}$$

where, from our discretization rules the action of the covariant finite difference operator D^+_μ on a site field $f(x)$ and a link field $f_\mu(x)$ are given explicitly by

$$\begin{aligned} D^+_\mu f(x) &= U_\mu(x)f(x+\mu) - f(x)U_\mu(x) \\ D^+_\mu f_\nu(x) &= U_\mu(x)f_\nu(x+\mu) - f_\nu(x)U_\mu(x+\nu)\,. \end{aligned}$$

The plaquette field $\mathcal{F}_{12}$ is thus given by

$$\mathcal{F}_{12}(x) = D^+_1 U_2(x) = U_1(x)U_2(x+1) - U_2(x)U_1(x+2)\,. \tag{5.3}$$

Notice that it is automatically antisymmetric in its indices and reduces to the usual Yang-Mills field strength in the continuum limit $a \to 0$. Using these rules we can now write down the form of the lattice gauge fermion in terms of component fields. The result, which is explicitly gauge invariant and reduces to the continuum expression eq. 2.1 in the naive continuum limit, is

$$\begin{aligned} S_L = \beta Q\,\mathrm{Tr}\sum_x \Bigg(& \frac{1}{4}\eta^\dagger(x)[\phi(x),\overline{\phi}(x)] + \chi_{12}^\dagger(x)\mathcal{F}_{12}(x) + \chi_{12}(x)\mathcal{F}_{12}(x)^\dagger + \\ & + \frac{1}{2}\chi_{12}^\dagger(x)B_{12}(x) + \frac{1}{2}\chi_{12}(x)B_{12}^\dagger(x) + \\ & + \frac{1}{2}\psi_\mu^\dagger(x)D_\mu^+\overline{\phi}(x) + \frac{1}{2}\psi_\mu(x)(D_\mu^+\overline{\phi}(x))^\dagger\Bigg)\,. \end{aligned} \tag{5.4}$$

This expression will also be Q-invariant if we can generalize the continuum twisted supersymmetry transformations in such a way that we preserve the property $Q^2 = \delta_G^\phi$. The following transformations do the job

$$\begin{aligned} QU_\mu &= \psi_\mu \\ Q\psi_\mu &= D_\mu^+\phi \\ Q\phi &= 0 \\ Q\chi_{12} &= B_{12} \\ QB_{12} &= [\phi,\chi_{12}]^{(12)} \\ Q\overline{\phi} &= \eta \\ Q\eta &= [\phi,\overline{\phi}] \end{aligned} \tag{5.5}$$

where the superscript notation indicates a *shifted* commutator

$$[\phi,\chi_{\mu\nu}]^{(\mu\nu)} = \phi(x)\chi_{\mu\nu}(x) - \chi_{\mu\nu}(x)\phi(x+\mu+\nu)\,. \tag{5.6}$$

These arise naturally when we consider the infinitesimal form of the gauge transformation property of the plaquette field. Notice that gauge invariance also dictates that we must use the covariant forward difference operator D_μ^+ on the right-hand side of the U_μ variation. The Q-transformations of the conjugate fields are similar

$$\begin{aligned} QU_\mu^\dagger &= \psi_\mu^\dagger \\ Q\psi_\mu^\dagger &= (D_\mu^+\phi)^\dagger \\ Q\chi_{12}^\dagger &= B_{12}^\dagger \\ QB_{12}^\dagger &= \left([\phi,\chi_{12}]^{(12)}\right)^\dagger\,. \end{aligned} \tag{5.7}$$

Carrying out the Q-variation leads to the following expression for the lattice action

$$\begin{aligned} S_L = \beta\,\mathrm{Tr}\sum_x \Bigg(& \frac{1}{4}[\phi(x),\overline{\phi}(x)]^2 - \frac{1}{4}\eta^\dagger(x)[\phi(x),\eta(x)] - \chi_{12}^\dagger(x)[\phi(x),\chi_{12}(x)]^{(12)} + \\ & + B_{12}^\dagger(x)B_{12}(x) + B_{12}^\dagger(x)\mathcal{F}_{12}(x) + B_{12}(x)\mathcal{F}_{12}(x)^\dagger + \\ & + \frac{1}{2}(D_\mu^+\phi(x))^\dagger D_\mu^+\overline{\phi}(x) + \frac{1}{2}D_\mu^+\phi(x)(D_\mu^+\overline{\phi}(x))^\dagger - \\ & - \chi_{12}^\dagger(x)D_1^+\psi_2(x) + \chi_{12}^\dagger(x)D_2^+\psi_1(x)) - \psi_2^\dagger(x)D_1^-\chi_{12}(x) + \end{aligned}$$

$$+ \psi_1^\dagger(x) D_2^- \chi_{12}(x) - \frac{1}{2}\psi_\mu^\dagger(x) D_\mu^+ \eta(x) - \frac{1}{2}\eta^\dagger(x) D_\mu^- \psi_\mu(x) + \\ + \psi_\mu^\dagger(x)[\overline{\phi}(x), \psi_\mu(x)]^{(\mu)}\Big) . \tag{5.8}$$

Notice in this expression the appearance of the covariant backward difference operator D_μ^- whose action on a plaquette field $f_{\mu\nu}$ is given explicitly by

$$D_\mu^- f_{\mu\nu}(x) = f_{\mu\nu}(x) U_\mu^\dagger(x+\nu) - U_\mu^\dagger(x-\mu) f_{\mu\nu}(x-\mu) \tag{5.9}$$

the resulting object transforming as a link field under gauge transformations. Similarly, following our discretization prescription, the lattice covariant difference operator D_μ^- acting on a link field yields

$$D_\mu^- f_\mu(x) = f_\mu(x) U_\mu^\dagger(x) - U_\mu^\dagger(x-\mu) f_\mu(x-\mu) \tag{5.10}$$

and transforms as a site field under gauge transformations. Notice that the Q-variation of $F_{12}(x)$ yields a derivative term on the anticommuting fields completely analogous to that seen in the continuum. In addition the (link) shifted commutator term $\psi_\mu^\dagger[\overline{\phi}, \psi_\mu]$ appears naturally from the variation of the last terms in the lattice gauge fermion. Finally we must integrate out the multiplier fields B_{12} and $B_{12}^\dagger$ resulting in the term

$$\beta \,\mathrm{Tr} \sum_x \mathcal{F}_{12}(x)^\dagger \mathcal{F}_{12}(x) . \tag{5.11}$$

This can be written

$$\beta \,\mathrm{Tr} \sum_x \left(2I - U_P - U_P^\dagger\right) + \beta \,\mathrm{Tr} \sum_x (M_{12} + M_{21} - 2I) \tag{5.12}$$

where

$$U_P = \mathrm{Tr}\left(U_1(x) U_2(x+1) U_1^\dagger(x+2) U_2^\dagger(x)\right) \tag{5.13}$$

resembles the usual Wilson plaquette operator and

$$M_{12}(x) = U_1(x) U_1^\dagger(x) U_2(x+1) U_2^\dagger(x+1) . \tag{5.14}$$

Notice that the second term vanishes when the gauge field is restricted to be unitary which is equivalent to requiring $\mathrm{Im} A_\mu^a(x) = 0$. In this case the action is nothing more than the usual Wilson gauge action and *does not* suffer from the vacuum degeneracy problem inherent in the models constructed in [19].

Having constructed the lattice action we must now discuss the path integral we will use to define the quantum theory. Initially this path integral will include integrations over both fields and their conjugates. To make contact with the continuum theory we would like to integrate along a contour on which the imaginary parts of all fields bar the scalars vanish (and the scalars are taken to be hermitian conjugates of each other). It is clear that the Yang-Mills action, the scalar action and the determinant resulting from integration over the twisted fermions are still gauge invariant when so restricted. Furthermore it is clear that the resulting action is real and positive definite (at least for small lattice spacing when the lattice action approaches the continuum action) for such a choice of contour. The only remaining question relates to the Q-symmetry - specifically do the the twisted supersymmetric Ward identities still hold when the lattice theory is restricted in this way ? To see that this is the case remember that the action is Q-exact and hence any supersymmetric Ward identity can be computed *exactly* in the limit $\beta \to \infty$. But in such a limit I can expand the gauge links to leading order in A_μ and recover the (complexified) continuum action and Q-transformations. Furthermore, it is known that the continuum theory can be consistently restricted to the contour we have described [12] and so we infer that the lattice Ward identities should also be satisfied on this contour.

Returing to the lattice action given in eq. 5.8 we may rewrite the fermionic pieces in the form

$$\overline{\Psi} M \Psi \tag{5.15}$$

where M the matrix operator can be written in block form

$$M = \begin{pmatrix} -[\phi,]^{(p)} & K \\ -K^\dagger & [\overline{\phi},]^{(p)} \end{pmatrix} \tag{5.16}$$

and the kinetic operator K is given by

$$K = \begin{pmatrix} D_2^+ & -D_1^+ \\ -D_1^- & -D_2^- \end{pmatrix} \tag{5.17}$$

with the spinors defined as

$$\overline{\Psi} = \left(\chi_{12}^\dagger, \eta^\dagger, \psi_1^\dagger, \psi_2^\dagger\right) \qquad \Psi = \begin{pmatrix} \chi_{12} \\ \eta \\ \psi_1 \\ \psi_2 \end{pmatrix}. \tag{5.18}$$

Notice that the form of the kinetic operator K ensures that the lattice theory does not exhibit spectrum doubling. The absence of doubles is not an accident but, as advertised, is a consequence of discretizing a purely geometrical action written in terms of Kähler-Dirac fields. Finally it is not possible at non-zero lattice spacing to cast the theory in terms of a single Dirac spinor. One easy way to see this is to recall that the components of the continuum Dirac spinor contain objects like $(\frac{1}{2}\eta + i\chi_{12})$. In the language of Kähler-Dirac fields such quantities arise after the self-dual projection. They are problematic on the lattice since they do not transform simply at finite lattice spacing under gauge transformations. Thus a reduction to a single Dirac spinor is not possible in the lattice theory. Instead after integrating out the anticommuting degrees of freedom along the contour $\mathrm{Im}\Psi^a = 0$ we will be left with a factor

$$\mathrm{Pf}(M)\,. \tag{5.19}$$

In the continuum limit we know that this pfaffian is equivalent to a real, positive definite determinant. Thus from the point of view of simulations it should be possible to replace the pfaffian by the expression

$$\mathrm{Pf}(m) = \det{}^{\frac{1}{2}} M \tag{5.20}$$

without encountering a sign problem for small enough lattice spacing.

6. Conclusions

In this paper we have derived a lattice action for $N = 2$ super Yang-Mills theory in two dimensions. We first show that the continuum form of the action can be written succinctly in the language of differential forms and Kähler-Dirac fields. This manifestly geometric starting point allows us to discretize the theory without inducing spectrum doubling and maintaining both gauge invariance and a single twisted supersymmetry. The lattice theory naturally contains complex fields — to access the correct continuum limit requires that an appropriate contour be chosen when evaluating the path integral. We argue that both gauge invariance and the twisted supersymmetry can be maintained if this contour is chosen such that the imaginary parts of all component field bar the scalars are taken to vanish. The scalars $\phi^a(x)$ and $\overline{\phi}^a(x)$ can be taken to be the complex conjugate of each other. The resulting fermion operator can be shown to be positive definite at least for small enough lattice spacing.

There are several directions for further work. The most obvious is the need for numerical simulations to check some of the conclusions of this work, perhaps most importantly, the claim that the twisted Ward identities are maintained along the contour required to define the path integral. It is also possible to generalize these ideas to four dimensions. The $N = 4$ theory contains 16 real supercharges which, with an appropriate twist, can be represented using a four dimensional (real) Kähler-Dirac field. Furthermore, it is possible to embed the bosonic degrees of freedom of this theory in another real Kähler-Dirac field with commuting components just as for the two dimensional theory. Derivation of the appropriate gauge fermion and corresponding twisted supersymmetry will be presented elsewhere [31]. Such a formulation should allow for discretization using the prescription described here. Secondly, the geometric nature of these theories should allow them to be formulated on arbitrary simplicial lattices [32, 33]. This would allow study of twisted super Yang-Mills theories on curved spaces. Summing over such simplicial lattices may provide a connection to (lattice regulated) supergravity theories.

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